REGULAR-SS11: On the Computational Complexity of Positive Linear Functionals on $\mathcal{C}[0;1]$*

Hugo Férée[1,2] and Martin Ziegler[1,3]

[1] Dept. of Mathematics, TU Darmstadt, Schlossgartenstr.7, D-64289 Darmstadt, Germany
[2] Université de Lorraine, LORIA, UMR 7503, Vandœuvre-lès-Nancy, F-54506, France
Inria, Villers-lès-Nancy, F-54600, France
CNRS, LORIA, UMR 7503, Vandœuvre-lès-Nancy, F-54506, France
[3] School of Computing, KAIST, Yuseong-gu 291 Daehak-ro, 34141 Daejeon, South Korea

Abstract. The Lebesgue integration has been related to polynomial counting complexity in several ways, even when restricted to smooth functions. We prove analogue results for the integration operator associated with the Cantor measure as well as a more general second-order **#P**-hardness criterion for such operators. We also give a simple criterion for relative polynomial time complexity and obtain a better understanding of the complexity of integration operators using the Lebesgue decomposition theorem.

1 Motivation and Introduction

Devising a complexity theory of higher-type computation is an ongoing endeavour since at least 25 years [Cook91, KaCo96, IBR01, KaCo10, FGH13, FeHo13, KSZ15]. Perhaps more modestly, we are interested in classifying the continuous linear functionals Ψ on the space $\mathcal{C}[0;1]$ of continuous functions on the real unit interval: first non-uniformly, that is, investigate the computational complexity of the real number $\Psi(f)$ for arbitrary but fixed polynomial-time computable $f \in \mathcal{C}[0;1]$; and then uniformly with (approximations, in some sense, to) f 'given' by means of oracle access, yet still for fixed Ψ.

According to the *Riesz–Markov–Kakutani Representation Theorem* precisely every positive (i.e. monotone) linear functional $\Psi : \mathcal{C}[0;1] \to \mathbb{R}$ is of the form $\Psi(f) = \int_0^1 f(t)\,d\nu(t)$ for some regular Borel measure ν on $[0;1]$. *Lebesgue's Decomposition Theorem* in turn asserts each such ν to admit a (unique) decomposition $\nu = \nu_\mathrm{d} + \nu_\mathrm{c} + \nu_\mathrm{s}$, where

i) ν_d is discrete,
ii) ν_c is absolutely continuous w.r.t. the canonical (*i.e.* Lebesgue) measure λ, and
iii) ν_s is singular continuous.

This theorem can be useful in the study of the computational complexity of an integration operator, by determining the complexity of each component.

Remark 1. i) The prototype of a discrete measure is Dirac's family δ_z with $\delta_z([a;b]) = 1$ if $z \in [a;b]$ and $\delta_z([a;b]) = 0$ otherwise. The induced positive linear functional is simply evaluation at z — and polynomial-time computable uniformly in z, essentially by definition.
More generally every discrete measure on $[0;1]$ has the form $\nu_\mathrm{d} = \sum_{j\in\mathbb{N}} \delta_{z_j} \cdot w_j$ for two sequences $(z_j) \subseteq [0;1]$ and $(w_j) \subseteq [0;\infty)$ with $\sum_j w_j < \infty$.

*Supported in part by the *Marie Curie International Research Staff Exchange Scheme Fellowship* 294962 within the 7th European Community Framework Programme and by the *German Research Foundation* (DFG) with project Zi 1009/4-1. A SHORT version of this work was presented at CCA 2015.

ii) The prototype of an absolutely continuous measure on $[0;1]$ is thus λ defined by $\lambda([a;b]) = b-a$; and the complexity of its induced positive linear functional on $\mathcal{C}[0;1]$, namely of definite Riemann integration, has been characterized as $\#\mathsf{P}_1$ (Fact 3a+b); and indefinite Riemann integration as $\#\mathsf{P}$ (Fact 3c+d). Moreover restricting to continuously differentiable argument does not reduce the worst-case complexity.
In general, according to the classical Radon–Nikodym Theorem, to every absolutely continuous measure ν_c on $[0;1]$ there exists some measurable $\varphi : [0;1] \to [0;\infty)$ such that $\int_0^x f(t)\,d\nu_c(t) = \int_0^x f(t)\varphi(t)\,dt$ holds for all $f \in \mathcal{C}[0;1]$ and $0 \le x \le 1$.

iii) The prototype of a singular continuous measure is Cantor's, that is, given by the Devil's Staircase or Cantor–Lebesgue–Vitali function $S : [0;1] \to [0;1]$ as cumulative distribution and inducing as functional the parametric Riemann-Stieltjes integral $(f,x) \mapsto \int_0^x f(t)\,dS(t)$.

1.1 Recap of Discrete, Real, and Second-Order Complexity Theory

We presume familiarity with discrete complexity theory and only briefly recall the classes

- P of decision problems $L \subseteq \Sigma^*$ to which membership "$\vec{u} \in L$?" is decidable within a number of steps polynomial in the input length $|\vec{u}|$;
- EXP of decision problems decidable in time bounded by some exponential polynomial in the input length;
- PSPACE of decision problems to which membership is decidable using an at most polynomial amount of memory;
- FP of total function problems $f : \Sigma^* \to \mathbb{N}$ computable in a number of steps polynomial in the input length with output encoded in binary;
- NP of decision problems $L \subseteq \Sigma^*$ of the following form for some $V \in \mathsf{P}$ and some integer polynomial p: $L = \{\vec{u} \mid \exists \vec{v} : |\vec{v}| \le p(|\vec{u}|),\ \langle \vec{u}, \vec{v}\rangle \in V\}$.
- $\#\mathsf{P}$ of counting (i.e. function) problems of the form

$$\psi : \Sigma^* \ni \vec{u} \mapsto \#\big(\{\vec{v} \mid |\vec{v}| \le p(|\vec{u}|),\ \langle \vec{u}, \vec{v}\rangle \in V\}\big) \in \mathbb{N}$$

- $\#\mathsf{P}_1$ of *unary* counting problems of the form

$$\psi_1 : \mathbb{N} \ni n \mapsto \#\big(\{\vec{v} \mid |\vec{v}| \le p(n),\ \langle 1^n, \vec{v}\rangle \in V\}\big) \in \mathbb{N}$$

with hierarchy $\mathsf{P} \subseteq \begin{matrix}\mathsf{P}^{\#\mathsf{P}_1}\\ \mathsf{NP}\end{matrix} \subseteq \mathsf{P}^{\#\mathsf{P}} \subseteq \mathsf{PSPACE} \subseteq \mathsf{EXP}$.
In particular recall that $\#\mathsf{P}$ may not be closed even under simple functions [HeOg02, §5.2]. Here Σ denotes some fixed finite alphabet containing at least symbols $\mathtt{0}$ and $\mathtt{1}$; and

$$\Sigma^* \times \Sigma^* \ni (\vec{v}, \vec{w}) \mapsto \langle \vec{v}, \vec{w}\rangle \in \Sigma^*$$

an injective polynomial-time computable string pairing function having polynomial-time decidable image and polynomial-time computable partial inverse.

Concerning the complexity of real functions we refer to [Ko91, Corollary 2.21]: Computing $f : [0;1] \to [0;1]$ within time $t(n)$ means (or rather is equivalent) to compute in the discrete sense of complexity $t(n)$ some function $\tilde{f} : \{\mathtt{0},\mathtt{1}\}^* \to \{\mathtt{0},\mathtt{1}\}^*$ such that it holds for $\mu(n) := t(n+1)+1$, for all $n \in \mathbb{N}$, and for all $\vec{u} \in \{\mathtt{0},\mathtt{1}\}^n$:

$$\forall x \in [0;1] : \quad \big|x - \mathrm{bin}(\vec{u})/2^{\mu(n)}\big| \ \le\ 2^{-\mu(n)} \quad \Rightarrow \ \big|f(x) - \mathrm{bin}\big(\tilde{f}(\vec{u})\big)/2^n\big| \ \le\ 2^{-n} \ . \qquad (1)$$

Here, for $\vec{v} \in \{\mathtt{0},\mathtt{1}\}^n$, $\mathrm{bin}(\vec{v})/2^{|\vec{v}|} := \sum_{j=0}^{n-1} v_j 2^{j-n} \in [0;1)$ is a dyadic rational. Again $\{\mathtt{0},\mathtt{1}\}$ can be replaced by any other at least binary alphabets without affecting the complexity more than polynomially. In the sequel for the Cantor distribution ternary rational approximations $\mathrm{tri}(\vec{u})/3^{|\vec{u}|} = \sum_{j=0}^{m-1} u_j 3^{j-m}$ for $\vec{u} \in \{\mathtt{0},\mathtt{1},\mathtt{2}\}^m$ will often turn out as convenient.

Recall that $\mu : \mathbb{N} \to \mathbb{N}$ satisfying $d(x,x') \leq 2^{-\mu(n)} \Rightarrow e\big(f(x), f(x')\big) \leq 2^{-n}$ is called a *modulus of continuity* of $f : X \to Y$ with metric spaces (X,d) and (Y,e). Concerning a uniform complexity of operators in analysis, we follow [KaCo10, §3] in letting $\mathsf{Pred} := \{\mathtt{0},\mathtt{1}\}^{(\{\mathtt{0},\mathtt{1}\}^*)}$ denote the set of all predicates on finite binary strings; $\mathsf{Reg} \subseteq \{\mathtt{0},\mathtt{1}\}^{**} := (\{\mathtt{0},\mathtt{1}\}^*)^{(\{\mathtt{0},\mathtt{1}\}^*)}$ the family of all (total) mapping $\varphi : \{\mathtt{0},\mathtt{1}\}^* \to \{\mathtt{0},\mathtt{1}\}^*$ that are length-monotonous in the sense of satisfying $|\varphi(\vec{u})| \leq |\varphi(\vec{v})|$ whenever $|\vec{u}| \leq |\vec{v}|$. In this case, the size function $|\varphi| : \mathbb{N} \ni |\vec{v}| \mapsto |\varphi(\vec{v})| \in \mathbb{N}$ of φ is well-defined. A *second-order polynomial* P is a term over $+, \times, \mathbb{N}$ and first-order variable n as well as second-order variable ℓ. An oracle Turing machine $\mathcal{M}^?$ *computes* a partial function $F :\subseteq \mathsf{Reg} \to \mathsf{Reg}$ when producing $F\big(\varphi\big)(\vec{v})$, given $\vec{v} \in \{\mathtt{0},\mathtt{1}\}^*$ and oracle access to $\varphi \in \mathrm{dom}(F)$. $\mathcal{M}^?$ runs in *second-order polynomial time* if it makes a number steps bounded by $P(|\varphi|, |\vec{v}|)$ for some second-order polynomial P. The following notions and bold-face complexity classes are from [Kawa11, Definitions 2.10–2.13]:

Definition 2. *a)* $\mathbf{P}$ *is the class of total* $F : \mathsf{Reg} \to \mathsf{Pred}$ *computable in second-order polynomial time.*

b) $\mathbf{FP}$ *is the class of total* $F : \mathsf{Reg} \to \mathsf{Reg}$ *computable in second-order polynomial time.*

c) $\mathbf{NP}$ *is the class of total* $G : \mathsf{Reg} \to \mathsf{Pred}$ *of the form*

$$G\big(\varphi\big)(\vec{v}) = \mathtt{1} \;\Leftrightarrow\; \exists \vec{w} \in \{\mathtt{0},\mathtt{1}\}^{P(|\varphi|,|\vec{v}|)} : F\big(\varphi\big)(\vec{v},\vec{w}) = \mathtt{1}$$

for some $F \in \mathbf{P}$ *and some second-order polynomial* P.

d) $\#\mathbf{P}$ *is the class of total* $G : \mathsf{Reg} \to \mathsf{Reg}$ *of the form*

$$G\big(\varphi\big)(\vec{v}) = \mathrm{bin}\left(\#\big\{\vec{w} \in \{\mathtt{0},\mathtt{1}\}^{P(|\varphi|,|\vec{v}|)} : F\big(\varphi\big)(\vec{v},\vec{w}) = \mathtt{1}\big\}\right)$$

for some $F \in \mathbf{P}$ *and some second-order polynomial* P.

e) For $F, G :\subseteq \mathsf{Reg} \to \mathsf{Reg}$*, a second-order polynomial-time (Weihrauch-) reduction from* F *to* G *is a triple* (U,V,W) *with* $U,V,W \in \mathbf{FP}$ *such that* $U(\varphi) \in \mathrm{dom}(G)$ *for every* $\varphi \in \mathrm{dom}(F)$ *and*

$$\forall \vec{v} \in \{\mathtt{0},\mathtt{1}\}^* : \quad F\big(\varphi\big)(\vec{v}) \;=\; W\Big(\varphi\Big)\Big\langle G\Big(U\big(\varphi\big)\Big)\big((V(\varphi))(\vec{v})\big), \vec{v}\Big\rangle$$

f) Some $\varphi \in \mathsf{Reg}$ encodes $f \in C([0;1],[0;1])$ *if it is of the form*

$$\varphi : \{\mathtt{0},\mathtt{1}\}^* \;\ni\; \vec{u} \;\mapsto\; \mathtt{1}^{\mu(n)}\,\mathtt{0}\;\mathrm{bin}\left(\tilde{f}(\vec{u})\right)$$

for some modulus of continuity μ *of* f *and* $\tilde{f}$ *according to Equation (1).*

g) $F :\subseteq \mathsf{Reg} \to \mathsf{Reg}$ represents *some operator* $\Lambda :\subseteq C([0;1],[0;1]) \to C([0;1],[0;1])$ *if it maps every encoding* $\varphi \in \mathsf{Reg}$ *of some* $f \in \mathrm{dom}(\Lambda)$ *to some encoding* $F(\varphi)$ *of* $\Lambda(f)$.

h) We may identify such an operator with the functional

$$\Lambda :\subseteq C([0;1],[0;1]) \times [0;1] \;\ni\; (f,x) \;\mapsto\; \Lambda\big(f\big)(x) \;\in\; [0;1] \;\;.$$

j) $\Lambda :\subseteq C([0;1],[0;1]) \to C([0;1],[0;1])$ *is computable in second-order polynomial-time if it admits a representative* $F :\subseteq \mathsf{Reg} \to \mathsf{Reg}$ *computable in second-order polynomial-time.*

Compare [BrGh11, HiPa13] for a computable version of Item e). We record that closure under composition of second-order polynomial-time computability yields transitivity of second-order polynomial-time reducibility. As opposed to the first-order complexity classes with the P/NP Millennium Prize and related open problems, the second-order versions are generally known distinct.

Fact 3 *a) If $f : [0;1] \to [0;1]$ is polynomial-time computable and $\#\mathsf{P}_1 \subseteq \mathsf{FP}_1$, then $\int_0^1 f(t)\,dt$ is again polynomial-time computable.*

b) There exists a polynomial-time computable smooth (i.e C^∞) $f : [0;1] \to [0;1]$ such that polynomial-time computability of $\int_0^1 f(t)\,dt$ implies $\#\mathsf{P}_1 \subseteq \mathsf{FP}_1$.

c) If $f : [0;1] \to [0;1]$ is polynomial-time computable and $\#\mathsf{P} \subseteq \mathsf{FP}$, then $[0;1] \ni x \mapsto \int_0^x f(t)\,dt$ is again polynomial-time computable.

d) There exists a polynomial-time computable smooth $f : [0;1] \to [0;1]$ such that polynomial-time computability of $[0;1] \ni x \mapsto \int_0^x f(t)\,dt$ implies $\#\mathsf{P} \subseteq \mathsf{FP}$.

e) For any $G :\subseteq \mathsf{Reg} \to \mathsf{Reg}$ representing (in the sense of Definition 2g) indefinite integration

$$\int : C([0;1],[0;1]) \ \ni\ f \ \mapsto\ \big([0;1]^2 \ni (x,y) \mapsto \int_x^y f(t)\,dt\big) \ \in\ C^1([0;1];[0;1]) \qquad (2)$$

there exists a $F \in \boldsymbol{\#\mathsf{P}}$ and a second-order polynomial-time reduction from G to F.

f) For every $F \in \boldsymbol{\#\mathsf{P}}$ and every $G :\subseteq \mathsf{Reg} \to \mathsf{Reg}$ representing the restriction $\int\big|_{C^\infty([0;1],[0;1])}$ there exists a second-order polynomial-time reduction from F to G.

For the first four items see [Ko91, Theorems 5.32+5.33]. They are non-uniform in that f is considered fixed and not as input. In other words, they only consider the image of a complexity class by the operator. This contrasts with the uniform Items e) and f), essentially [Kawa11, Theorem 4.21], where the complexity of the operator itself is considered.

1.2 Overview, Techniques, and Related Work

The present work investigates the non-uniform and uniform computational complexity of other (types of) positive linear functionals on $C[0;1]$ and $C^\infty[0;1]$. Similarly to Fact 3, Sections 2 and 3) relate Cantor integration non-uniformly and uniformly equivalent to $\#\mathsf{P}_1$, $\#\mathsf{P}$, and $\boldsymbol{\#\mathsf{P}}$. Perhaps surprisingly, it is thus as hard as ordinary/absolutely continuous Riemann integration. (Along the way we prove the Devil's Staircase S to be computable in polynomial time.)

On the other hand, Example 16 constructs singular continuous measures that does render integration polynomial-time computable — after Subsection 4.1 identifying classes of measures for which integration is $\boldsymbol{\#\mathsf{P}}$–hard. Conversely, Example 14 constructs a discrete measure rendering integration $\boldsymbol{\#\mathsf{P}}$–hard — based on Subsection 4.2 devising classes of measures for which integration is polynomial-time computable.

Proof techniques are essentially refinements and variations of those employed in [Ko91, §5.4] and [Kawa11]: On the one hand encoding a polynomial-time decidable verifier $V \subseteq \{\mathtt{0},\mathtt{1}\}^* \times \{\mathtt{0},\mathtt{1}\}^*$ as polynomial-time computable (smooth) real function f_V consisting of infinitely many 'steps' (cmp. Figures 1 and 2) such that the hard discrete counting problem $\#\{\vec{y} : (\vec{x},\vec{y}) \in V\}$ can be recovered from approximations of the continuous integral over f_V w.r.t. the measure under consideration; and on the other hand expressing approximations to said integral as discrete counting problem with polynomial-time decidable verifier;

and uniformly analyzing the 'reduction' $V \mapsto f_V$ as well as its converse in terms of second-order polynomial-time complexity theory. In fact jointly scaling the steps in x-direction and y-direction is a delicate trade-off: such as to (i) recover discrete arguments $\vec{x} \in \{0,1\}^*$ from approximations to real arguments $x \in [0;1]$ as well as (ii) recover discrete results $\#\{\vec{y} : (\vec{x},\vec{y}) \in V\}$ from approximations to the real values $\int f_V(t)\,dS(t)$ while (iii) maintaining continuity, smoothness, and polynomial-time computability of f_V; cmp. Remark 18b).

Regarding more general but qualitative computability investigations of measures the reader may refer for instance to [Schr07, HRW12, MTY14, Coll14].

2 Smooth Cantor Integration is at Least as Hard as Continuous Riemann

Proposition 4. *a) Cantor's Function $S : [0;1] \to [0;1]$ is Hölder-continuous with exponent $\alpha = \ln(2)/\ln(3)$ and computable within polynomial time.*

b) For every interval $I = [a;b] \subseteq \mathbb{R}$ and every non-decreasing continuous $g : I \to \mathbb{R}$ it holds $\int_{g(a)}^{g(b)} f(t)\,dt = \int_I f\big(g(s)\big)\,dg(s)$.

Proof. a) Recall that S is the uniform limit of a sequence of piecewise linear functions defined inductively by $S_0 := \mathrm{id} : [0;1] \to [0;1]$ and $S_{n+1}(t) := \begin{cases} S_n(3t)/2 & \text{if } t \leq \frac{1}{3} \\ 1/2 & \text{if } \frac{1}{3} \leq t \leq \frac{2}{3} \\ S_n(3t-2)/2 + 1/2 & \text{if } \frac{2}{3} \leq t \end{cases}$

More precisely $\|S_{n+1} - S_n\|_\infty \leq \|S_n - S_{n-1}\|_\infty/2$, hence $\|S_n - S\|_\infty \leq 2^{-n}$. Note that S_n has $\sum_{j=1}^n 2^j = 2^{n+1} - 2$ breakpoints at certain triadic rational points $t \in \mathbb{T}_n := \mathbb{Z}/3^n$. Moreover the restriction $S_n\big|_{\mathbb{T}_n} \to \mathbb{D}_n := \mathbb{Z}/2^n$ is well-defined and uniformly computable in time polynomial in n. According to (a minor adaptation of) [Ko91, Theorem 2.22], S is therefore computable in polynomial time.

b) Consider the generalized Darboux sums

$$U\big((s_j), f\circ g, g\big) = \sum\nolimits_j \sup\nolimits_{s\in[s_j,s_{j+1}]} f\big(g(s)\big)\cdot\big(g(s_{j+1}) - g(s_j)\big),$$
$$L\big((s_j), f\circ g, g\big) = \sum\nolimits_j \inf\nolimits_{s\in[s_j,s_{j+1}]} f\big(g(s)\big)\cdot\big(g(s_{j+1}) - g(s_j)\big)$$

by hypothesis both converging (from above and below, respectively) to $\int_I f\big(g(s)\big)\,dg(s)$, where (s_j) denotes a partition of I. Substituting $t_j := g(s_j)$ thus yields a partition of $g(I)$ with classical Darboux sums $U\big((t_j), f, \mathrm{id}\big) = U\big((s_j), f\circ g, g\big)$ and $L\big((t_j), f, \mathrm{id}\big) = L\big((s_j), f\circ g, g\big)$ converging to $\int_{g(I)} f(t)\,dt$; and vice versa. □

It follows that $\mathcal{C}[0;1] \ni f \mapsto f\circ S \in \mathcal{C}[0;1]$ and $\mathcal{C}^1[0;1] \ni f \mapsto f\circ S \in C^{0,\alpha}[0;1]$ are well-defined reductions from Riemann to Cantor integration computable within second-order polynomial time. Applied to Friedman and Ko's polynomial-time computable $f \in C^\infty([0;1],[0;1])$ with $\#\mathsf{P}_1$–'complete' integral, one obtains a polynomial-time computable Hölder-continuous $h : [0;1] \to [0;1]$ such that $\int_0^1 h(t)\,dS(t)$ is not computable in polynomial time unless $\#\mathsf{P}_1 \subseteq \mathsf{FP}$.

Note that $f\circ S$ is not differentiable in general. Moreover the reduction seems restricted to definite integration (and thus only achieves $\#\mathsf{P}_1$–hardness rather than $\#\mathsf{P}$) since the Cantor integration bounds a and b cannot computably be recovered from the Riemann ones $S(a)$ and $S(b)$ even in the multivalued sense.

Cantor integration is as hard as Riemann integration in the non-uniform and the uniform senses.

Theorem 5. *a) There exists a polynomial-time computable* smooth *(i.e. infinitely often differentiable) $h : [0;1] \to [0;1]$ such that $\int_0^1 h(t)\,dS(t)$ is not computable in polynomial time unless $\#\mathsf{P}_1 \subseteq \mathsf{FP}$.*

b) There exists a polynomial-time computable smooth $h : [0;1] \to [0;1]$ such that $[0;1]^2 \ni (a,b) \mapsto \int_{\min\{a,b\}}^{\max\{a,b\}} h(t)\,dS(t)$ is not computable in polynomial time unless $\#\mathsf{P} \subseteq \mathsf{FP}$.

c) For every $F \in \#\mathbf{P}$ and every $G :\subseteq \mathsf{Reg} \to \mathsf{Reg}$ representing the definite Cantor integration operator on smooth arguments, that is, the mapping

$$C^\infty([0;1],[0;1]) \ \ni \ f \ \mapsto \ \Big([0;1]^2 \ni (a,b) \mapsto \int_{\min\{a,b\}}^{\max\{a,b\}} f(t)\,dS(t) \in [0;1]\Big) \ \in \ C^{0,\alpha}([0;1]^2,[0;1])$$

there exists a second-order polynomial-time reduction from F to G.

Proof. These proofs are inspired from the proof of the hardness of the Riemann integration in [Kawa11].

a) Let g be a $\#\mathsf{P}_1$ function. By definition, there exists a polynomial time computable function g_1 and a polynomial P such that $\forall n \in \mathbb{N}, g(\mathtt{1}^n) = \#\{\vec{v} \in \{\mathtt{0},\mathtt{1}\}^{P(n)} : g_1(\mathtt{1}^n, \vec{v}) = 1\}$.
Recall that the Cantor set is the intersection over $n \in \mathbb{N}$ of unions of 2^n disjoint intervals of length 3^{-n} and that the Cantor integral of a constant function over one of these intervals (which we will call *Cantor intervals* in the following) is equal to this constant multiplied by the size of the interval.
We will define a smooth function h such that the values of g_1 will be encoded as the (constant) value of h on one of these intervals.
First, assume we have a smooth polynomial time computable function *step* as described by figure 1. We will define the function h such that its values on the Cantor interval

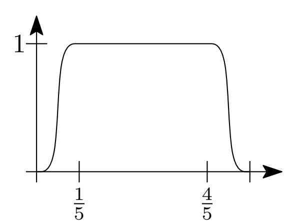

Fig. 1. The "smooth step function" *step*.

$I_n = [3^{-n} - 3^{-(n+1)}; 3^{-n}]$ will encode the values of $g_1(n, \vec{v})$ for $|\vec{v}| = P(n)$.
More precisely, we divide this interval into $2^{P(n)}$ smaller Cantor-intervals $[l_{n,\vec{v}}; r_{n,\vec{v}}]$ of width $3^{-(P(n)+n+1)}$:

$$\begin{cases} r_{n,\vec{v}} = & 3^{-n} - 3^{-(n+1)} \sum_{i=0}^{P(n)-1} 2 \cdot v_i 3^{i-P(n)} \\ l_{n,\vec{v}} = & r_{n,\vec{v}} - 3^{-(P(n)+n+1)} \end{cases},$$

where v_i is the i^{th} bit of $\vec{v}$.
Note that these intervals are pairwise disjoint (for all n and $\vec{v}$), and we even have that the open intervals $(l_{n,\vec{v}} - 3^{-(P(n)+n+2)}; r_{n,\vec{v}} + 3^{-(P(n)+n+2)})$ are also pairewise disjoint.
Now, if $g_1(\mathtt{1}^n, \vec{v}) = 1$, we define h on $[l_{n,\vec{v}} - 3^{-(P(n)+n+2)}; r_{n,\vec{v}} + 3^{-(P(n)+n+2)}]$ as the the *step* function horizontally scaled to this interval and vertically scaled by $3^{-(P(n)\cdot n+n+1)}$.

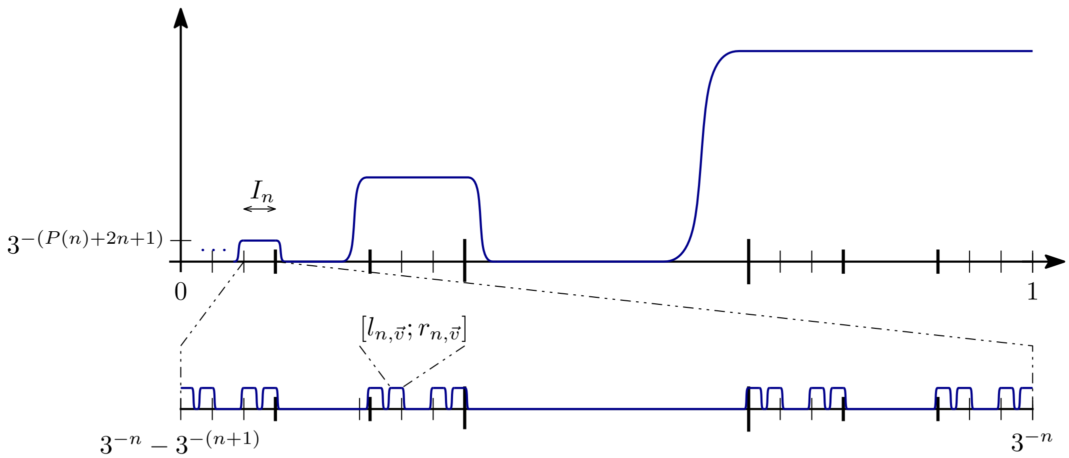

Fig. 2. The upper part of the figure show the envelope of h based on the intervals I_n. The lower part is a more precise envelope based on the subdivision of I_n into $P(n)$ intervals.

The function h is equal to 0 everywhere else. Figure 2 illustrates the pattern behind this construction.

Note that due to the smoothness of *step*, and since the intervals do not overlap, h is also smooth on $(0;1]$. Also, since the step function is scaled vertically faster (by a factor 3^n) than horizontally when going to 0, h is also smooth on $[0;1]$. Now, this function is constant on every interval $[l_{n,\vec{v}}; r_{n,\vec{v}}]$ and its value depends on the value of $g_1(\mathtt{1}^n, \vec{v})$. More precisely, the Cantor integral of h over such an interval is equal to $3^{-(P(n)\cdot(n+1)+2n+2)} \cdot g_1(\mathtt{1}^n, \vec{v})$. Since the Cantor integral is equal to 0 outside such intervals, the integral $\int_{I_n} h(t)\, dS(t)$ is equal to the sum over $\vec{v}$ ($|\vec{v}| = P(n)$) of such integrals, that is $g(\mathtt{1}^n) \cdot 3^{-(P(n)\cdot(n+1)+2n+2)}$ by definition of g_1.

This function is also polynomial time computable, since given $x \in [0;1]$ and a precision p, and $x < 3^{-p}$ then we can output 0. Otherwise, we can determine in polynomial time if x is close (up to $3^{-(P(n)+n+2)}$) to an interval $[l_{n,\vec{v}}; r_{n,\vec{v}}]$, in which we have to compute $g_1(\mathtt{1}^n, \vec{v})$ and the value of the corresponding *step* function in polynomial time if it is equal to 1.

The Cantor integral $\int_0^1 h(t)\, dS(t)$ is equal to $\sum_{n\in\mathbb{N}} g(\mathtt{1}^n) 3^{-(P(n)\cdot(n+1)+2n+2)}$, so if we assume (without loss of generality) that $P(n) = \Omega(n^2)$, then its ternary expansion is equal to the concatenation of the ternary encoding of $g(\mathtt{1}^n)$ for $n \in \mathbb{N}$ (with padding zeroes). More precisely, the $P(n)$ least significant digits of a tryadic number approaching this integral with precision $3^{-(P(n)\cdot(n+1)+3n+2)}$ is equal to $g_1(\mathtt{1}^n)$. So if this integral is a polynomial time computable real number, then g is a polynomial time computable first-order function, and thus $\#\mathsf{P}_1 \subseteq \mathsf{FP}$.

b) Assume we already proved c) and let g a $\#\mathsf{P}$ problem. In particular it is in $\boldsymbol{\#\mathsf{P}}$ and c) implies that (using the definition of the second-order Weihrauch-reduction) there are $U, V_1, V_2, W \in \mathbf{FP}$ (with no first-order parameter φ since g is a first-order function) such that: $\forall \vec{u} \in \{0;1\}^*, g(\vec{u}) = W\left(\int\limits_{V_1(\vec{u})}^{V_2(\vec{u})} U(t)\, dS(t), \vec{u}\right)$.

The function U is then polynomial time computable, and smooth according to the proof of c). If $(a,b) \mapsto \int_a^b U(t)\, dS(t)$ is polynomial time computable, then so would g, which implies $\#\mathsf{P} \subseteq \mathsf{FP}$.

c) Let $F \in \#\mathsf{P}$ be the second-order counting function associated with the polynomial function F_0 and the second-order polynomial P. We first describe a polynomial time computable function U which computes a smooth function from an input φ for F_0. The construction of $U(\varphi)$ will be similar to the one of h in the proof of *a)*. More precisely given ϕ, we divide the interval I_n into 2^n Cantor intervals $I_{\vec{v}} = [l_{n,\vec{v}}; r_{n,\vec{v}}]$, with $\vec{v} \in \{0,1\}^n$. Each of these intervals is then divided in the same way into $2^{P(|\varphi|,n)}$ Cantor intervals $I_{\vec{v},\vec{w}}$, with $\vec{w} \in \{0,1\}^{P(|\varphi|,n)}$. Similarly as in *a)*, $U(\varphi)$ is defined on such an interval (and on a small neighborhood) as either 0 or the smooth step function scaled by $3^{-(P(n)+2n+1)}$ horizontally, and $3^{-(P(n)\cdot n+2n+1)}$ vertically, depending on the value of $F_0(\varphi,\vec{v},\vec{w})$.
This function is smooth, and (second-order) polynomial time computable from φ. In addition, its Cantor integral over the interval $I_{\vec{v}}$ is equal to $3^{P(|\vec{v}|)\cdot(|\vec{v}|+1)+4|\vec{v}|+2} \cdot F(\varphi,\vec{v})$. Then, we define $V_1(\vec{v})$ and $V_2(\vec{v})$ as the endpoints of $I_{\vec{v}}$, which are indeed polynomial time computable with respect to $|\vec{v}|$. Finally, given φ and $\vec{v}$, the function $W(x,\vec{v})$ computes a $3^{-(P(|\vec{v}|)\cdot(|\vec{v}|+1)+4|\vec{v}|+3)}$-approximation of x, multiplies it by $3^{P(|\vec{v}|)\cdot(|\vec{v}|+1)+4|\vec{v}|+2}$, and outputs the closest integer value. Altogether, we obtain that for all φ and $\vec{v}$, $F(\varphi,\vec{v}) = W(\int_{V_1(\vec{v})}^{V_2(\vec{v})} U(\varphi)(t)\,dS(t),\vec{v})$, which describes a second-order polynomial reduction from F to the definite Cantor integration. □

3 Continuous Cantor Integration is at Most as Hard as Smooth Riemann

Reducing the problem of approximating $\int_0^s f(t)\,dS(t)$ up to error 2^{-n} to that of approximating $\int_0^s g_n(t)\,dt$ for some smooth g_n is easy: Since the Cantor measure concentrates all weight to 2^n subintervals $I_{n,k}$ of $[0;1]$ while neglecting the complementing ones, define f_n to be zero on the latter and otherwise equal to f cut and 'squeezed' into the $I_{n,k}$. This f_n is only piecewise continuous but can be approximated up to L^1–error 2^{-n} by a smooth one — depending on n. Based on the following result, Corollary 17 will yield some g independent of n — at the expense of certain 'post-processing' the integral's value.

Theorem 6. *a) Let $f:[0;1]\to[0;1]$ be computable in polynomial time and suppose $\#\mathsf{P}_1 \subseteq \mathsf{FP}$. Then the definite Cantor integral over f, that is the real number $\int_0^1 f(t)\,dS(t)$, is again computable in polynomial time.*

b) Let $f:[0;1]\to[0;1]$ be computable in polynomial time and suppose $\#\mathsf{P} \subseteq \mathsf{FP}$. Then the indefinite Cantor integral over f, that is the mapping $[0;1]^2 \ni (x,y) \mapsto \int_{\min\{x,y\}}^{\max\{x,y\}} f(t)\,dS(t)$, is again computable in polynomial time.

c) For every $G :\subseteq \mathsf{Reg} \to \mathsf{Reg}$ representing the mapping

$$C([0;1],[0;1]) \ni f \mapsto \Big([0;1]^2 \ni (x,y) \mapsto \int_{\min\{x,y\}}^{\max\{x,y\}} f(t)\,dS(t) \in [0;1]\Big) \in C^{0,\alpha}([0;1]^2,[0;1])$$

there exists a second-order polynomial-time reduction from G to some $F \in \#\mathsf{P}$.

In-/definite Cantor integration is thus at most as hard as Riemann integration (and, equivalently, $\#\mathsf{P}$).

Proof. a) Let $\mu:\mathbb{N}\to\mathbb{N}$ be a polynomial modulus of continuity of f and, modifying Equation (1) as indicated, $\tilde{f}:\{0,1,2\}^*\to\{0,1\}^*$ computable in polynomial time such that

$$x \in [0;1] \wedge \big|x - \mathrm{tri}(\vec{u})/3^{\mu(n)}\big| \;\le\; 3^{-\mu(n)} \quad\Rightarrow\; \big|f(x) - \mathrm{bin}(\vec{v})/2^n\big| \;\le\; 2^{-n} \;\; .$$

Then the following function $\psi_1 : \{\mathtt{1}\}^* \to \mathbb{N}$ belongs to $\#\mathsf{P}_1$:

$$\psi_1(\mathtt{1}^n) \;:=\; \#\big\{(\vec{w},\vec{v}) \in \{\mathtt{0},\mathtt{2}\}^{\mu(n)} \times \{\mathtt{0},\mathtt{1}\}^n :\; \mathrm{bin}\big(\tilde{f}(\vec{w})\big) \geq \mathrm{bin}(\vec{v})\big\}$$

Now the Cantor distribution assigns weight $1/2^m$ to each interval $\big[\frac{\mathrm{tri}(\vec{w})}{3^m};\frac{\mathrm{tri}(\vec{w})+1}{3^m}\big]$, $\vec{w} \in \{\mathtt{0},\mathtt{2}\}^m$. Moreover f varies by at most 2^{-n} on each such interval for $m := \mu(n)$. Therefore $\psi_1(\mathtt{1}^n)/2^{n+\mu(n)}$ is a Darboux sum approximating $\int_0^1 f(t)\,dS(t)$ up to error 2^{1-n}.

b) Similarly to a), but now take into account triadic approximations $\mathrm{tri}(\vec{a})/3^{\mu(n)}$ to $\min\{x,y\}$ and $\mathrm{tri}(\vec{b})/3^{\mu(n)}$ to $\max\{x,y\}$ in the $\#\mathsf{P}$–function $\psi(\mathtt{1}^n,\vec{a},\vec{b}) :=$

$$\#\big\{(\vec{w},\vec{v}) \in \{\mathtt{0},\mathtt{2}\}^{\mu(n)} \times \{\mathtt{0},\mathtt{1}\}^n :\; \mathrm{tri}(\vec{a}) \leq \mathrm{tri}(\vec{w}) \leq \mathrm{tri}(\vec{b}),\; \mathrm{bin}\big(\tilde{f}(\vec{w})\big) \geq \mathrm{bin}(\vec{v})\big\}$$

c) Consider $H : \mathsf{Reg} \to \mathsf{Reg}$, defined by $H\big(\varphi\big)(\langle \mathtt{1}^n,\vec{a},\vec{b}\rangle,\langle\vec{w},\vec{v}\rangle) := 1$ if

$$\vec{v} \in \{\mathtt{0},\mathtt{1}\}^n,\; \vec{a},\vec{b} \in \{\mathtt{0},\mathtt{1},\mathtt{2}\}^{\mu(n)},\; \vec{w} \in \{\mathtt{0},\mathtt{2}\}^{\mu(n)},\; \mathrm{tri}(\vec{a}) \leq \mathrm{tri}(\vec{w}) \leq \mathrm{tri}(\vec{b}),\; \mathrm{bin}\big(\tilde{f}(\vec{w})\big) \geq \mathrm{bin}(\vec{v})$$

for $\mathtt{1}^{\mu(n)}\,\mathtt{0}\,\mathrm{bin}\big(\tilde{f}(\vec{w})\big) := \varphi(\vec{w})$, and $H\big(\varphi\big)(\langle \mathtt{1}^n,\vec{a},\vec{b}\rangle,\langle\vec{w},\vec{v}\rangle) := 0$ otherwise. Then obviously $H \in \mathbf{P}$ holds, and hence $F \in \#\mathbf{P}$ for

$$F\big(\varphi\big)(\langle \mathtt{1}^n,\vec{a},\vec{b}\rangle) \;=\; \mathrm{bin}\Big(\#\big\{(\vec{w},\vec{v}) \in \{\mathtt{0},\mathtt{2}\}^{\mu(n)} \times \{\mathtt{0},\mathtt{1}\}^n \;:\; \mathrm{tri}(\vec{a}) \leq \mathrm{tri}(\vec{w}) \leq \mathrm{tri}(\vec{b}), \\ \mathrm{bin}\big(\tilde{f}(\vec{w})\big) \geq \mathrm{bin}(\vec{v}),\; \mathtt{1}^{\mu(n)}\,\mathtt{0}\,\mathrm{bin}\big(\tilde{f}(\vec{w})\big) := \varphi(\vec{w})\big\}\Big)$$

satisfying $\Big|F\big(\varphi\big)(\langle \mathtt{1}^n,\vec{a},\vec{b}\rangle)/2^{n+\mu(n)} \;-\; \int_{\mathrm{tri}(\vec{a})/3^{|\vec{a}|}}^{\mathrm{tri}(\vec{b})/3^{|\vec{b}|}} f(t)\,dS(t)\Big| \;\leq\; 2^{-n}$ according to b). □

4 Generalized hardness and tractability conditions

4.1 Hardness

The analysis of the similarities between the proofs of uniform $\#\mathbf{P}$-hardness of Lebesgue and Cantor integrations gives a list of simple criteria, which can be applied to more cases.

Theorem 7. *Let μ be a measure over $[0;1]$ such that there for every second-order polynomial P, there are rational nonempty open intervals I_w^f and $I_{w,w'}^f$ computable in uniform second-order polynomial time, where $f \in \mathbb{N} \to \mathbb{N}$, $w, w' \in \Sigma*$, and $|w'| \leq P(|w|)$, such that:*

a) $w_1 \neq w_2 \implies I_{w_1}^f \cap I_{w_2}^f = \emptyset$
b) $w_1' \neq w_2' \implies I_{w,w_1'}^f \cap I_{w,w_2'}^f = \emptyset$
c) $I_{w,w'}^f \subseteq I_w^f$
d) The function

$$(\mathbb{N} \to \mathbb{N}) \times \mathbb{D} \ni f, d \mapsto \begin{cases} \langle w, w' \rangle & \text{if } d \in I_{w,w'}^f \\ \varepsilon & \text{otherwise} \end{cases}$$

is second-order polynomial time computable.
e) There exists m_w^f polynomial time computable with respect to $|f|$ and $|w|$ such that

$$1 \leq m_w^f \cdot \int_{I_{w,w'}^f} s_{I_{w,w'}^f}\, d\mu \leq 1 + 2^{-P(f,|w|)}$$

where $s \in \mathcal{C}[0;1]$ is any polynomial time computable function such that $s(0) = s(1) = 0$.

Then for every **#P** *function F, there exists a second-order polynomial time reduction from F to some $G :\subseteq \mathsf{Reg} \to \mathsf{Reg}$ representing the definite μ integration. In addition, if s is smooth and vanishes at 0 and 1, then this can be restricted to integration of smooth functions.*

Proof. Let $F \in$ **#P** and F_0 be the second-order counting function associated with the second-order polynomial P_0. Given an input oracle φ, we define a continuous function $U(\varphi)$ this way:

$$U(\varphi)(x) = \begin{cases} |I^f_{w,w'}|^{|w|} s_{I^f_{w,w'}}(x) & \text{if } x \in I^f_{w,w'} \text{ and } F_0(\varphi, w, w') = 1 \\ 0 & \text{otherwise.} \end{cases}$$

where $s_{(a;b)}(x) = \frac{s(\frac{x-a}{b-a})}{b-a}$ and $f = P_0(|\varphi|) + 1$.

First, U is well-defined, since the intervals $I^f_{w,w'}$ are pairewise disjoint. It also has a polynomial time computable rational approximation function: given a rational q and a precision n, decide in polynomial time (using d) in which interval q is (and output 0 if it is in none). Then, $|I^f_{w,w'}|^{|w|} s_{I^f_{w,w'}}(x)$ can be computed in polynomial time, since s and the endpoints of $I^f_{w,w'}$ are. It is also easy to see that the modulus of continuity of the function $x \mapsto |I^f_{w,w'}|^{|w|} s_{I^f_{w,w'}}(x)$ is the same as the one of s, and thus $U(\varphi)$ has a polynomial modulus of continuity (even independent from φ). Altogether, this proves that U is polynomial time computable.

Secondly, if s is smooth, then so is $U(\varphi)$. Indeed, the k^{th} derivative of $U(\varphi)$ at $x \in I^f_{w,w'}$ is equal to $|I^f_{w,w'}|^{|w|-(k+1)} \cdot s^{(k)}(\frac{x-a}{b-a})$ where $I^f_{w,w'} = (a;b)$. Now, if (x_n) converges to $x \in [0;1]$, then either

- x_n is infinitely many times in a given interval $I^f_{w,w'}$. Since it is open, it is eventually in this interval, in wich case $U(\varphi)^{(k)}(x_n)$ converges to $U(\varphi)^{(k)}(x_n)$ by continuity of $U(\varphi)^{(k)}$ on $I^f_{w,w'}$ (by smoothness of s);
- otherwise x_n can not be infinitely many times in more that two such intervals since they do not intersect. In this case, x is one of the endpoints of such intervals and $U(\varphi)(x_n)$ converges to $0 = U(\varphi)(x)$ (since $s(0) = s(1) = 0$);
- otherwise, x_n is eventually outside the union of the intervals (*i.e.* in a closed set, so where x also belongs) and $U(\varphi)(x_n) = 0 = U(\varphi)(x)$;
- finally, x_n can be decomposed into a sequence outside any interval (whose image by $U(\varphi)^{(k)}$ converges to 0), or in an interval $I^f_{w_n,w'n}$ occurring only finitely many times. This implies that the sequence w_n diverges to $+\infty$ (since there are only a finite number of w' for a given w) and thus $U(\varphi)^{(k)}$ also converges to 0 on this subsequence (since $s^{(k)}$ is bounded, and $|I^f_{w,w'}|^{|w|-(k+1)}$ converges to 0). This is indeed equal to $U(\varphi)^{(k)}(x)$, otherwise we would be in the first case.

Finally, $F(\varphi, w)$ can be indeed computed in polynomial time from the μ-integral of U. Indeed, according to hypothesis e,

$$F(\varphi, w) = \sum_{|w'| \le P_0(|\varphi|,|w|)} 1 \le \sum_{|w'| \le P_0(|\varphi|,|w|)} m_w^{|\varphi|} \cdot \int_{I^{P_0(|\varphi|,|w|)}_{w,w'}} s_{I^{P_0(|\varphi|,|w|)}_{w,w'}} d\mu \le F(\varphi, w) \cdot (1 + 2^{-(P_0(|\varphi|,|w|)+1)}),$$

and since $F(\varphi, w) \le 2^{P_0(|\varphi|,|w|)}$, and that the sum of the integrals is equal to the integral over I^f_w, we obtain:

$$F(\varphi, w) \le m_w^{|\varphi|} \cdot \int_{I_w^{P_0(|\varphi|,|w|)}} U(\varphi) d\mu \le F(\varphi, w) + \frac{1}{2}.$$

In other words we can define a second-order polynomial time computable function $W(\varphi, g, w)$ which computes a $\frac{1}{2}$-approximation of $m_w^{|\varphi|}$ multiplied by the real number represented by g and outputs the closest integer. In this case, we obtain:

$$F(\varphi, w) = W(\varphi)(G(U(\varphi))(I_w^{P_0(|\varphi|,|w|)}), w),$$

if G represents the definite μ-integration. □

4.2 Tractability

Conversely, there is some simple sufficient condition for a positive linear operator to be polynomial time computable with respect to an oracle. For this, we will use the main result of [FGH13], where the authors define the sets of relevant points $(R_n)_{n\in\mathbb{N}}$ of a real norm on $\mathcal{C}[0;1]$. Roughly speaking, R_n is the set of points of $[0;1]$ where it is sufficient to know an input 1-Lipschitz function $f \in \mathcal{C}[0;1]$ in order to determine its norm with precision 2^{-n} (see the original article for a precise definition). The theorem states that polynomial time (relatively to an oracle) computable real norms are exactly those which depend on a 'small' set of points in this sense:

Definition 8. *A set A can be polynomially covered, if there exists a polynomial P such that for all $n \in \mathbb{N}$, A can be covered by $P(n)$ balls of radius 2^{-n}. In other words, A has metric entropy* $\log \circ P$.

Fact 9 ([FGH13]) *A real norm can be computed in polynomial time relatively to an oracle if and only if its sets of relevant points $(R_n)_{n\in\mathbb{N}}$ can be polynomially covered uniformly in n.*

Even if an integration operator is not a norm, it can be completed into one, so that we can apply a weak form of the previous theorem.

Theorem 10. *If the support of a measure ν can be polynomially covered, then the corresponding indefinite integration operator $(x, y, g) \mapsto \int_x^y g d\nu$ is computable in polynomial time with respect to an oracle.*

Proof. Let ν be such a measure, with support S. There exists a polynomial time computable norm F over $\mathcal{C}[0;1]$. For $x \leq y$ in $[0;1]$, the operators $G^+_{x,y}(f) = F(f)+\int_x^y f^+ d\nu$ and $G^-_{x,y}(f) = F(f) + \int_x^y f^- d\nu$ (where f^+ and f^- are the positive and negative parts of f) are norms. We need to separate the positive and negative parts in order to make these operators always positive.

The set of relevant points R_n of ν-integration is included in the closure of S. Indeed, the integral of any 1-Lipschitz function defined on a neighborhood of $x \notin \bar{S}$ is equal to zero as soon as this neighborhood does not intersect S, by definition of the support of a measure.

Thus, for all n, the set of relevant points of $G^+_{x,y}$ and $G^-_{x,y}$ are included in $\bar{S} \cup R_n^F$, where $(R_n^F)_n$ are the relevant sets of F. By application of Fact 9 to F, $(R_n^F)_n$ can be polynomially covered uniformly in n, and since it is also the case for S, and thus for its closure, it is true for the union.

By application of the other implication of Fact 9, these norms are polynomial time computable with respect to an oracle. In fact, this is also true for the corresponding operators G^+ and G^-, uniformly in x and y. Since the indefinite ν-integration operator is equal to $G^+ - G^-$, it is also polynomial time computable with respect to an oracle, and allows us to conclude. □

Even though allowing an arbitrary oracle may seem powerful, such operators are still weaker than Lebesgue or Cantor integrals.

Corollary 11. *The indefinite integration operator associated with such a measure is not* **#P***-hard.*

Indeed, such an operator would allow to compute Lebesgue or Cantor integration operator in relative polynomial time, which is impossible (in particular by an application of Fact 9); cmp. also [KaPa14].

4.3 Applications and examples

First, let us focus on the absolutely continuous case, *i.e.* where the integral of a function f is equal to the Lebesgue integral of $f \cdot g$, where g is a measurable function.

The simplest non-trivial example is the Lebesgue integration (with $g = 1$) is already **#P**-hard, which makes us believe that this is the case in general. It is already the case if g is polynomial time computable.

Proposition 12. *Let $g \in \mathcal{C}[0;1]$ be polynomial time computable and non-identically zero. Then $(x, y, f) \mapsto \int_x^y f(t) \cdot g(t)dt$ is* **#P***-hard.*

Proof. Let g be such a function. Since g is computable, it is continuous and thus there exists $k \in \mathbb{N}$ such that $2^{-k} < g$ on an interval $[a; b]$, $a < b$ (if not, it is the case for $-g$). Also, we can find an appropriate step function such that its integral α on $[0; 1]$ is polynomial time computable, and $2^{-l} \leq \alpha$ for some $l \in \mathbb{N}$. Since g is polynomial time computable, it has a polynomial modulus of continuity m_g.

Let P be a second-order polynomial, and f a first-order function. It is possible to define I_w as a sub-interval of $[a; b]$ of width at most $2^{-m_g(P(f,|w|)+k+2)}$ and define the intervals $I_{w,w'}$ as uniform subdivisions of this interval.

Since g is computable in polynomial time (with respect to f and $|w|$), it is possible to compute lower and upper bound c and d of g on I^f_w, such that $d - c \leq 2^{-(P(f,|w|)+k+2)}$ by definition of the modulus of continuity. Thus we have that $\frac{d}{c} \leq 1 + \frac{(d-c)}{a} \leq 2^{-(P(f,|w|)+2)}$.

Similarly, we can compute in polynomial time lower and upper bounds c' and d' of α with precision $2^{-(P(f,|w|)+l+2)}$, and thus, $\frac{d'}{c'} \leq 2^{-(P(f,|w|)+2)}$.

Finally, we have that

$$\frac{c \cdot c'}{|I^f_{w,w'}|^2} \leq \int_{I^f_{w,w'}} s_{I^f_{w,w'}} \cdot g d\lambda \leq \frac{d \cdot d'}{|I^f_{w,w'}|^2},$$

that is to say $1 \leq \dfrac{|I^f_{w,w'}|^2}{c \cdot c'} \cdot \displaystyle\int_{I^f_{w,w'}} s_{I^f_{w,w'}} \leq (1 + 2^{-(P(f,|w|)+2)})^2 \leq 1 + 2^{-P(f,|w|)}$. □

However, we don't know if this still holds for functions g with higher complexity. Intuitively, it seems that we need g to be polynomial time computable in order to retrieve some information about f from its integral (see hypothesis e of Theorem 7).

Now we can have a look at the case of discrete measures, *i.e.* corresponding to posive linear operators F of the form: $F(f) = \sum_{n\in\mathbb{N}} \alpha_n \cdot f(\beta_n)$, where $\alpha_n > 0$ and $\beta_n \in [0; 1]$. It is not surprising that when this sum is finite, then the integration operator F is polynomial time computable with respect to an oracle (where an appropriate oracle encodes the α_i's and β_i's). But Theorem 10 even gives a more general result.

Proposition 13. *If F is a discrete integration operator of the form $F(f) = \sum_{n\in\mathbb{N}} \alpha_n \cdot f(\beta_n)$, such that the set $B = \{\beta_i \mid i \in \mathbb{N}\}$ can be polynomially covered, then F is computable in relative polynomial time.*

Proof. In this case, the support of F is contained in the closure of B, which is thus can also be polynomially covered, and Theorem 10 applies. □

Conversely, there are discrete measures defining **#P**-hard integration operators.

Example 14. Let $F(f) = \sum_{w\in\{0,1\}^*} 2^{-2|w|} f(\overline{0.w.1}^2)$, where $\bar{w}^2$ is the real number with binary expansion w. Its sequence of scaling factors decreases exponentially slowly, whereas its set of evaluation points covers all the dyadic rational numbers of the open interval $(0,1)$. This discrete positive linear operator is not computable in (relative) polynomial time. Moreover, we can apply Theorem 7 and deduce that it is **#P**-hard.

The conditions of the two theorems are not always necessary and there are cases where none of them apply. But it seems that most of the time, a result can still be obtained using the general shape of discrete measures.

Example 15. Let $F(f) = \sum_n f(d_n)$, where (d_n) is the standard enumeration of the dyadic rational numbers of $[0;1]$. It is an integration operator relative to a discrete measure whose support is $[0;1]$. Since an interval can't be polynomially covered, we can not apply Theorem 10. However, a direct application of [FGH13] or straightforward analysis allows us to prove that it is polynomial time computable. Indeed, to compute $F(f)$, it is sufficient to compute $f(d_0), \dots, f(d_{\mu(n)})$, if f has modulus of continuity μ.

Finally, the last case is the one of singular continuous measures. It is the hardest one, since there is no simple characterization of such measures.

We have already seen with the Cantor measure that an integration operator for such a measure can be **#P**-hard. But a similar measure can also be polynomial time computable.

Example 16. The Cantor set is defined by the intersection of sets $(C_n)_{n\in\mathbb{N}}$, where C_n is the union of 2^n disjoint intervals of size 3^{-n}. If we define a Cantor-like set $C' = \bigcap_{n\in\mathbb{N}} C'_n$, where C'_n is the intersection of 2^{-n} intervals of size $3^{2^{-n}}$ (*i.e.* exponentially smaller), then the corresponding measure has support C', which can be polynomially covered. By a direct application of Theorem 10, the associated positive linear operator is polynomial time computable with respect to an oracle. If in addition the endpoints of these intervals are polynomial time computable uniformly in n, then it is even simply polynomial time computable.

5 Conclusion and Perspectives

We have completed the complexity-theoretic classification of the three 'prototypes' of positive linear functionals on $\mathcal{C}[0;1]$: evaluation (discrete) is polynomial-time computable whereas both Riemann (absolutely continuous) and Cantor (singular continuous) integration both correspond to the discrete complexity class $\#\mathsf{P}_1$. More precisely they are uniformly second-order polynomial-time equivalent in the following sense:

Corollary 17. *a) There exists a second-order polynomial-time computable operator $U : C([0;1],[0;1]) \to C^\infty([0;1];[0;1])$, and second-order polynomial-time computable functionals $V_1, V_2, W : C([0;1],[0;1]) \times [-1;1] \to [0;1]$ such that the following holds:*

$$\forall f \in C([0;1],[0;1]) \ \ \forall 0 \le a \le b \le 1 : \quad \int_a^b f(t)\, dS(t) \ = \ W\Big(f, \int_{V_1(f,a)}^{V_2(f,b)} U\big(f\big)(t)\, dt\Big) \ .$$

b) There exists a second-order polynomial-time computable operator $U : C([0;1],[0;1]) \to C^\infty([0;1];[0;1])$ and second-order polynomial-time computable functionals $V_1, V_2, W : C([0;1],[0;1]) \times [0;1] \to [0;1]$ such that the following holds:

$$\forall f \in C([0;1],[0;1]) \ \ \forall 0 \le a \le b \le 1 : \quad \int_a^b f(t)\, dt \ = \ W\Big(f, \int_{V_1(f,a)}^{V_2(f,b)} U\big(f\big)(t)\, dS(t)\Big) \ .$$

Proof. Combine the second-order polynomial-time reductions of (smooth) Cantor integration to and from **#P** according to Theorems 5c) and 6c) with the known second-order polynomial-time reductions of (smooth) Riemann integration to and from **#P**. ☐

Remark 18. a) [Kawa11, Theorems 4.18+4.21] originally have asserted maximization and integration of bivariate functions

$$C([0;1]^2) \ \ni \ f \ \mapsto \ \big([0;1] \ni s \mapsto \max\{f(s,t) : 0 \le t \le 1\}\big) \ \in \ C[0;1] \tag{3}$$

$$C([0;1]^2) \ \ni \ g \ \mapsto \ \big([0;1] \ni s \mapsto \int_0^1 g(s,t)\, dt\big) \ \in \ C[0;1] \tag{4}$$

to be **NP**–complete and **#P**–complete, respectively. Since **NP** trivially reduces to **#P**, transitivity yields the existence of a second-order polynomial-time reduction from maximization (3) to integration (4): which seems quite surprising.

b) Fact 3e+f) refers to a univariate variant of Equation (4) with varying lower and upper integration bounds. In fact maximization in Equation (3) even remains **NP**–complete when only varying the upper bound, that is, the operator

$$C([0;1]) \ \ni \ f \ \mapsto \ \big([0;1] \ni s \mapsto \max\{f(t) : 0 \le t \le s\}\big) \ \in \ C[0;1]$$

Since such a reduction according to Definition 2e) is permitted only one invocation of the integration operator, we wonder whether also integration remains **#P**–complete with only upper bound varying: $C([0;1]) \ \ni \ g \ \mapsto \ \big([0;1] \ni y \mapsto \int_0^y f(t)\, dt\big) \ \in \ C([0;1])$.

5.1 Prototype vs. the General Case

Our investigation of the complexity of positive linear functionals Ψ on $\mathcal{C}[0;1]$ has focused on prototypical examples of each of the three basic types according to Riesz–Markov–Kakutani. For instance the d-dimensional Poisson problem has been shown [KSZ14] to boil down to absolutely continuous integration (ii). It is, however, easy to find Ψ that are harder than these prototypes: for example evaluation (i) at some EXP–complete point $z \in [0;1]$. This leads to

Question 19. Is there an integrable $g : [0;1] \to (0;\infty)$ such that $\int_0^1 f(t) \cdot g(t)\, dt$ is polynomial-time computable for every polynomial-time computable $f \in \mathcal{C}[0;1]$ even in case $\mathsf{P} \neq \mathsf{NP} \neq \mathsf{P}^{\#\mathsf{P}} \neq \mathsf{PSPACE}$?

Restricted to continuous g the answer is negative: Each such admits distinct rational (and in particular polynomial-time computable) $a,b\in[0;1]$ and $k\in\mathbb{N}$ with $1/k\leq|g|\leq k$ on $[a;b]$; w.l.o.g. $g=|g|$ there. Record that the polynomial-time computable smooth $h:[0;1]\to[0;1]$ with $\#\mathsf{P}_1$–'complete' $\int_0^1 h(t)\,dt$ can be achieved to vanish (with all derivatives) on $(-\infty;0]\cup[1;\infty)$; cmp. [Ko91, THEOREM 5.32d]. Scaling $f(t):=h\big(\frac{t-a}{b-a}\big)$ thus is still smooth on $[0;1]$ and polynomial-time computable; yet approximating

$$(b-a)\cdot k\cdot\int_0^1 f(t)\cdot g(t)\,dt \quad\in\quad \Big[\int_0^1 h(t)\,dt\ ;\ k^2\cdot\int_0^1 h(t)\,dt\Big]$$

up to error $2^{-n}/k^2$ recovers $\int_0^1 h(t)\,dt$ up to error 2^{-n}. This leaves it to look for integrable, nowhere essentially bounded g; cmp. `http://math.stackexchange.com/questions/620959`.

References

[BrGh11] V. BRATTKA, G. GHERARDI: "Weihrauch degrees, omniscience principles and weak computability", pp.143–176 in *J. Symbolic Logic* vol.**76:1** (2011).

[Coll14] P. COLLINS: "Computable Stochastic Processes", `arXiv:1409.4667` (2014).

[Cook91] S. COOK: "Computability and complexity of higher type functions", pp.51–72 in *Logic from Computer Science* (Y.N. Moschovakis, ed.), Springer (1991).

[FeHo13] H. FÉRÉE, M. HOYRUP: "Higher-order complexity in analysis", in *Proc. 10th Int. Conf. on Computability and Complexity in Analysis* (CCA'2013).

[FGH13] H. FÉRÉE, W. GOMAA, M. HOYRUP: "Analytical properties of resource-bounded real functionals", pp.647–671 in *Journal of Complexity* vol.**30:5** (2014).

[Frie84] H. FRIEDMAN: "The Computational Complexity of Maximization and Integration", pp.80–98 in *Advances in Mathematics* vol.**53** (1984).

[HeOg02] L.A. HEMASPAANDRA, M. OGIHARA: *The Complexity Theory Companion*, Springer (2002).

[HiPa13] K. HIGUCHI, A. PAULY: "The degree structure of Weihrauch-reducibility", pp.1–17 in *Logical Methods in Computer Science* vol.**9:2** (2013).

[HRW12] M. HOYRUP, C. ROJAS, K. WEIHRAUCH: "Computability of the Radon-Nikodym derivative", pp.1–11 in *Computability* vol.**1** (2012).

[IBR01] R. IRWIN, B. KAPRON, J. ROYER: "On Characterizations of the Basic Feasible Functionals, Part I", pp.117–153 in *Journal of Functional Programming* vol.**11** (2001).

[KaCo96] B.M. KAPRON, S.A. COOK: "A New Characterization of Type-2 Feasibility", pp.117–132 in *SIAM Journal on Computing* vol.**25:1** (1996).

[KaCo10] A. KAWAMURA, S.A. COOK: "Complexity Theory for Operators in Analysis", pp.495–502 in *Proc. 42nd Ann. ACM Symp. on Theory of Computing* (STOC 2010);
full version in *ACM Transactions in Computation Theory* vol.**4:2** (2012), article 5.

[KaPa14] A. KAWAMURA, A. PAULY: "Function Spaces for Second-Order Polynomial Time", pp.245–254 in *Proc. 10th Conf. on Computability in Europe*, Springer LNCS vol.**8493** (2014).

[Kawa11] A. KAWAMURA: "Computational Complexity in Analysis and Geometry", Dissertation, University of Toronto (2011).

[Ko91] K.-I. KO: "*Computational Complexity of Real Functions*", Birkhäuser (1991).

[MTY14] T. MORI, Y. TSUJII, M. YASUGI: "Computability of Probability Distributions and Characteristic Functions", *Logical Methods in Computer Science* vol.**9:3** (2013).

[Schr07] M. SCHRÖDER: "Admissible Representations of Probability Measures", pp.61–78 in *Electronic Notes in Theoretical Computer Science* vol.**167** (2007).

[KSZ14] A. KAWAMURA, F. STEINBERG, M. ZIEGLER: "Complexity of Laplace's and Poisson's Equation", abstract p.231 in *Bulletin of Symbolic Logic* vol.**20:2** (2014); full version to appear in *Logical Methods in Computer Science*.

[KSZ15] A. KAWAMURA, F. STEINBERG, M. ZIEGLER: "Computational Complexity Theory for classes of integrable functions", presented at the *JAIST Logic Workshop Series* (2015).

[Weih00] K. WEIHRAUCH: "*Computable Analysis*", Springer (2000).

[Weih03] K. WEIHRAUCH: "Computational Complexity on Computable Metric Spaces", pp.3–21 in *Mathematical Logic Quarterly* vol.**49:1** (2003).